The Problem

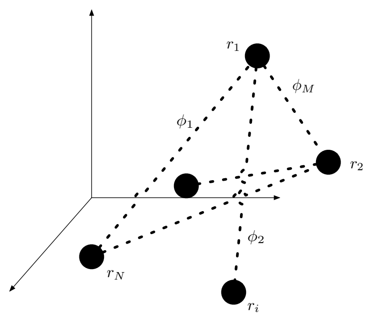

Figure 1: Balls and Sticks.

If we are given a set of N balls connected by M sticks, is the resulting object rigid (in however many dimensions)?

The Setup

Let $\{r_i\}_{i=1}^N$ represent the three-dimensional positions of the balls (all of this can be generalized to higher dimension, I take three). The connectivity is determined from a picture (or for ambitious interrogators, a three-dimensional model) – we can specify the sticks in terms of the balls they connect, and then linearly order in any manner we like.

To each stick, we assign a constraint: the stick connecting r_i to r_j can be described by its length

$$\phi_i = \|r_j - r_k\|^2 = (x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2 = d_{jk}^2 \tag{1}$$

The point is, we will take a "test move" and see which types of motions respect the constraint. For ϕ_i written above, if we move the j^{th} ball from r_j to $r_j + \delta r_j$, and the k^{th} ball to $r_k + \delta r_k$, then:

$$\begin{aligned} \delta\phi_i &= (r_j - r_k) \cdot (\delta r_j - \delta r_k) + O(\delta r^2) \\ &= (r_j - r_k) \cdot \delta r_j - (r_j - r_k) \cdot \delta r_k + O(\delta r^2) \end{aligned} \tag{2}$$

which implies a matrix multiplication. Define $\delta X \in \mathbb{R}^{3N}$ and the i^{th} row of the matrix J via

$$\begin{aligned} \delta X &\doteq \begin{pmatrix} \delta x_1 & \delta y_1 & \delta z_1 & \delta x_2 & \ldots & \delta x_N & \delta y_N & \delta z_N \end{pmatrix}^T \\ J_i &\doteq \begin{pmatrix} \ldots & \ldots & x_j - x_k & y_j - y_k & z_j - z_k & \ldots & x_k - x_j & y_k - y_j & z_k - z_j & \ldots \end{pmatrix} \end{aligned} \tag{3}$$

with all $\ldots = 0$ in the definition of J, and the first non-zero entries occurring at the $3(j-1)+1$ position, the second at $3(k-1)+1$ as indicated by the second line of (2). The matrix $J \in \mathbb{R}^{M\times(3N)}$ has one row for each of the "sticks".

If we are to have $\delta\phi_i = 0$ for all M constraints, then the matrix equation

$$\delta\phi = J \cdot \delta X = 0 \tag{4}$$

defines the condition that none of the constraint lengths change under the "motion" defined by δX (to first order). This tells us that $\delta X \in \text{Null}(J)$.

The Solution

Rigid body motion really means that the only degrees of freedom are global translations and rotations. To answer the original question posed here, we need to know the dimension of the null space of J. The object is rigid iff $\text{Dim}\,(\text{Null}(J)) = 6$. For *anything*, any combination of balls and sticks, there are automatically six degrees of freedom that leave internal lengths invariant: three translations and three rotations. That the null space must have dimension six in both directions (I mean "if and only if" here) for the object to be rigid is then clear – rigid bodies have six degrees of freedom corresponding to rotations and translations, and non-rigid bodies have these six and more.

The full SVD gives more information than $\text{Null}(J)$, but in theory, for $J = U\Sigma V^T$ (or whichever decorations of ^{T} you like), all one needs to do is count the zeros in the diagonal Σ to determine rigidity.

For a rigid body, one can also get the basis vectors for the six-dimensional null space from V^T. Here we are using more of the SVD information, and we can show that this six-dimensional space is in fact spanned by rotations and translations. Take the matrix:

$$Q \doteq \begin{pmatrix} T_x & T_y & T_z & R_x & R_y & R_z \end{pmatrix} \tag{5}$$

with $T_x \in \mathbb{R}^{3N}$ a column vector representing x-translation

$$T_x \doteq \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & \dots \end{pmatrix}^T \tag{6}$$

and similarly for T_y and T_z (for the rotations, it suffices to consider infinitesimal rotation generators). Then if V_{TR} is a matrix with six columns for the null space, you can calculate: $QQ^T V_{TR} - V_{TR} = 0$ to establish containment (and v.v. to get the other direction).

Example

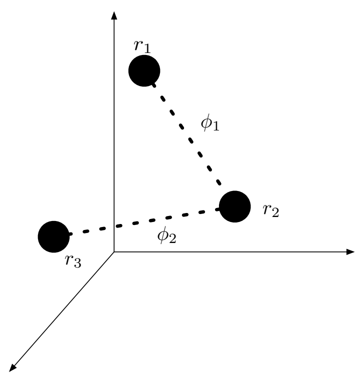

Figure 2: An example structure.

To be explicit about construction of J, we have for the example figure shown above:

$$J \doteq \begin{pmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 & 0 & 0 & 0 & x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ 0 & 0 & 0 & x_2 - x_3 & y_2 - y_3 & z_2 - z_3 & x_3 - x_2 & y_3 - y_2 & z_3 - z_2 \end{pmatrix} \tag{7}$$

and if I take $r_1 = (1, 0, 0)$, $r_2 = (0, 1, 0)$, $r_3 = (0, 0, 1)$, then

$$J \doteq \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix} \tag{8}$$

Mathematica presents me with two non-zero singular values for this matrix, the dimension of the (right-hand) space is 9, so I infer that the dimension of the null space is 7, this is not a rigid structure. Mathematica also has a function "NullSpace", of course, which must use the SVD internally.